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B.Sc. Part-I, Paper-I

Theory of Equations (Some Problems)

Q1) Find the condition that $x^3 - px^2 + qx - r = 0$ may have the sum of its roots zero.

Soln:- Let α, β, γ be the roots of $x^3 - px^2 + qx - r = 0$

Then $\alpha + \beta + \gamma = p$ ——— (1)

$\alpha\beta + \beta\gamma + \gamma\alpha = q$ ——— (2)

$\alpha\beta\gamma = +r$ ——— (3)

Given $\alpha + \beta = 0$ ($\because$ Sum of two roots is zero)

From (1), $\gamma = p$

$\because \gamma$ is a root of $x^3 - px^2 + qx - r = 0$

$\therefore \gamma^3 - p\gamma^2 + q\gamma - r = 0$, But $\gamma = p$

$\Rightarrow p^3 - p^3 + pq - r = 0 \Rightarrow pq - r = 0$

$\therefore pq = r$ is the required Condition.

Q2) Given that the roots of $x^3 + 3px^2 + 3qx + r = 0$ are in (i) A.P., Show that $2p^2 - 3qp + r = 0$ (ii) G.P., Show that $p^3 r = q^3$ (iii) H.P., Show that $2q^3 = r(3pq - r)$.

Soln:- Given equation is $x^3 + 3px^2 + 3qx + r = 0$

(i) The roots are in A.P. Let they are $a-d, a, a+d$.

Then $(a-d + a + a+d) = -3p \Rightarrow 3a = -3p \Rightarrow a = -p$ —(1)

and $a^3 + 3pa^2 + 3qa + r = 0 \Rightarrow$ [$\because a$ is the root of eqn]

But $a = -p \Rightarrow -p^3 + 3p(-p)^2 + 3q(-p) + r = 0$

$\Rightarrow -p^3 + 3p^3 - 3qp + r = 0$

$\Rightarrow 2p^3 - 3pq + r = 0$; proved.

(ii) The roots are in G.P.

Suppose the roots be $\frac{a}{R}, a, aR$

Given $\left(\frac{a}{R}\right)(a)(aR) = -r \Rightarrow a^3 = -r \Rightarrow a = (-r)^{1/3}$

Since, 'a' is a root of given equation

$\Rightarrow \{(-r)^{1/3}\}^3 + 3p\{(-r)^{1/3}\}^2 + 3q(-r)^{1/3} + r = 0$

$\Rightarrow -r + 3pr^{2/3} - 3qr^{1/3} + r = 0$

$\Rightarrow 3pr^{2/3} = 3qr^{1/3} \Rightarrow p\frac{r^{2/3}}{r^{1/3}} = q$

$\Rightarrow pr^{1/3} = q \Rightarrow p^3 r = q^3$ is the reqd. cond.; proved.

(iii) The roots of $x^3 + 3px^2 + 3qx + r = 0$ are in H.P. — (1)

Let $y = \frac{1}{x}$ so that $\frac{1}{y^3} + 3p\frac{1}{y^2} + 3q\frac{1}{y} + r = 0$ are in A.P. — (2)

Suppose $a-d, a, a+d$ be the roots of (2)

Then $[a-d+a+a+d] = -\frac{3q}{r} \Rightarrow 3a = -\frac{3q}{r}$

$\Rightarrow a = -\frac{q}{r}$ — (A)

$\because$ a is roots of $ry^3 + 3qy^2 + 3py + 1 = 0$

$\Rightarrow ra^3 + 3qa^2 + 3pa + 1 = 0$, But $a = -\frac{q}{r}$

$\Rightarrow r\left(-\frac{q}{r}\right)^3 + 3q\left(-\frac{q}{r}\right)^2 + 3p\left(-\frac{q}{r}\right) + 1 = 0$

$\Rightarrow -\frac{q^3}{r^2} + \frac{3q^3}{r^2} - \frac{3pq}{r} + 1 = 0$

$\Rightarrow -q^3 + 3q^3 - 3pqr + r^2 = 0$

$\Rightarrow 2q^3 - r(3pq - r) = 0$

$\Rightarrow 2q^3 = r(3pq - r)$ is the required condition

. proved.